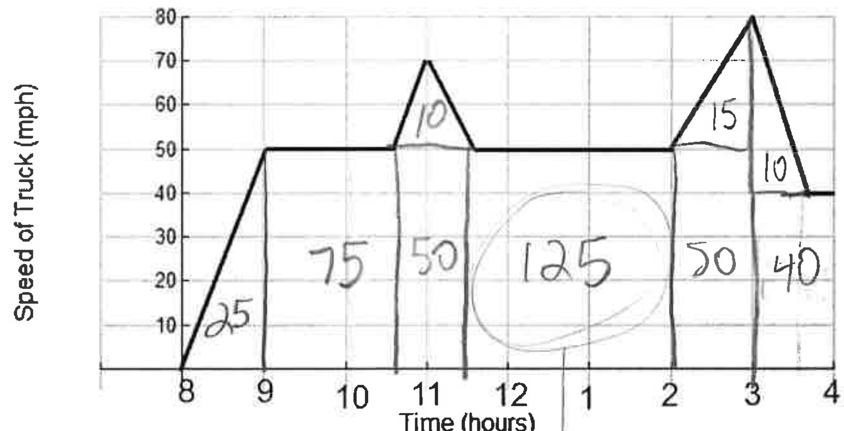
Relating Velocity and Distance Traveled
Consider the following problem that is seen in typical elementary school textbooks:

A train travels along a track at a steady rate of 75 mph from 7:00 AM to 9:00 AM. What is the total distance traveled by the train?

Being as mathematically sophisticated as we are, we know that *distance = rate x time* and can easily solve this problem → $75\frac{\text{miles}}{\text{hour}}\cdot 2\text{ hours} = 150\text{ miles}$. So how does this relate to calculus?

Take a look at the below graph of the velocity of a truck over time from 8:00 AM to 4:00 PM (the truck driver drove non-stop for 8 hours . . .)
Assume all non-lattice point intersections are half-hours.



1. How far did the truck travel between 11:30AM and 2PM?

$\frac{50\text{ miles}}{\cancel{\text{hr}}} * 2.5\ \cancel{\text{hrs}} = \boxed{125\text{ miles}}$

2. You may have ~~find~~ found your answer algebraically (using $D = r \cdot t$). How can you use the graph to find your answer *geometrically?* **Spend a few minutes being sure you understand how to find the answer geometrically – this is a BIG deal!!**

★ Find the area under the graph!

3. Use the method you developed in problem (2) to find the total distance traveled by the truck from 8 AM to 9AM.

Area $\Delta = \frac{1}{2}bh = \frac{1}{2}\cdot 1 \cdot 50 = \boxed{25\text{ miles}}$

OVER →

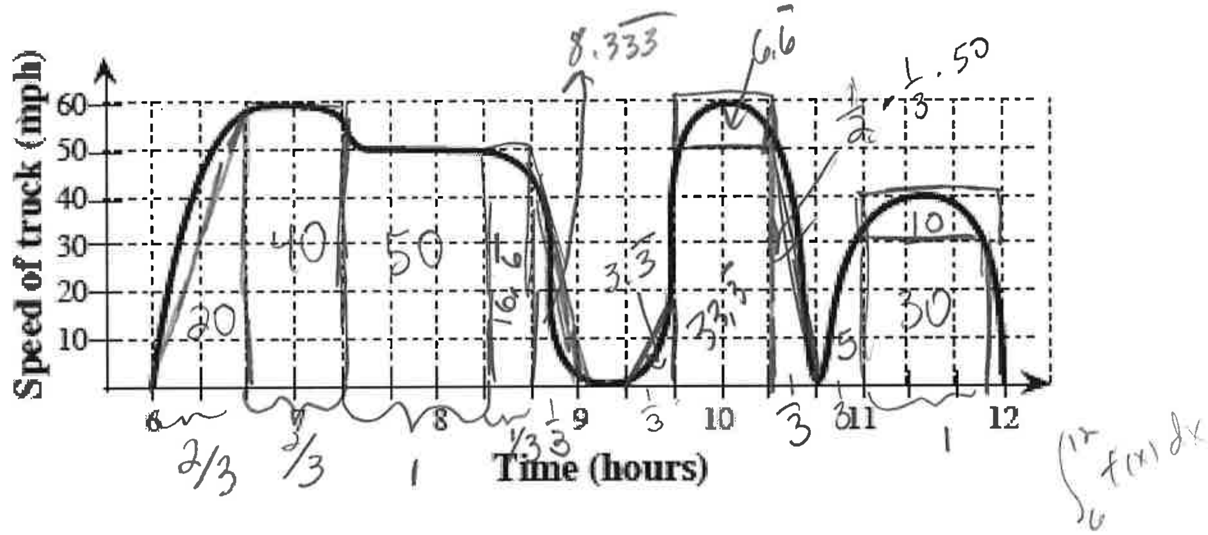
4. Find the total distance traveled from 8AM to 11:30AM

$25 + 75 + 60 = \boxed{160 \text{ miles}}$

5. Find the total distance traveled from 8AM to 4PM

$25 + 75 + 50 + 10 + 12.5 + 50 + 15 + 10 + 20 + 20 =$ $\boxed{400 \text{ miles}}$

Realistically, the change in velocity over time for trucks moving in the real world is not as linear as the pieces in the above graph. Below is a more realistic graph for the velocity over time by a truck from 6AM to 12PM.



6. Find (estimate) the distance traveled by the truck from 6AM to 9AM. Show your work on the above graph.

$\approx 20 + 40 + 50 + 16.\overline{6} + 8.\overline{3} = \boxed{135 \text{ miles}}$

7. Find (estimate) the total distance traveled by the truck from 6AM to 12PM.

$\approx 20 + 40 + 50 + 16.\overline{6} + 8.\overline{3} + 3.\overline{3} + 33.\overline{3} + \frac{25}{3} + 5 + 30 + 10$

$= \boxed{225 \text{ miles}}$

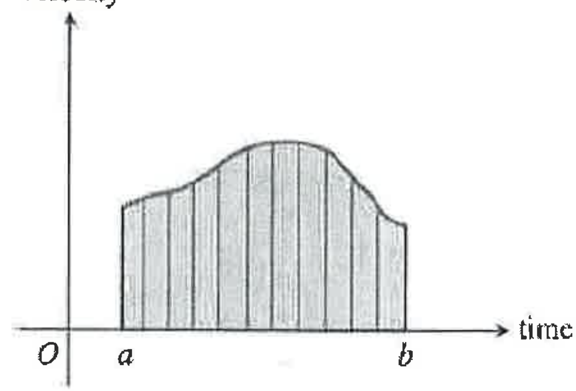
Notes:

- Area under a velocity curve gives the distance traveled

- Distance traveled is same as position for $v(t) > 0$

Does the area of an irregular region still give the total distance traveled over a time interval? Newton and Leibniz (and others before them) thought it would and that is why they were interested in a calculus for finding areas under curves. They imagined the time interval being partitioned into many tiny subintervals, each one so small that the velocity over it would essentially be constant. Geometrically, this was equivalent to slicing the irregular region into narrow strips, each of which would be nearly indistinguishable from a narrow rectangle.

The region to the right is partitioned into vertical strips that, if narrow enough, are almost indistinguishable from rectangles. Newton and Leibniz argued that, just as the total area could be found by summing the areas of the (essentially rectangular) strips, the total distance traveled could be found by summing the small distances traveled over the tiny intervals.

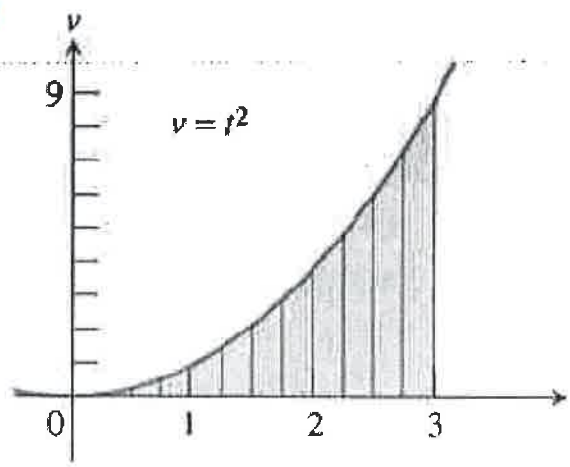


Example

A particle starts at $x = 0$ and moves along the x-axis with velocity $v(t) = t^2$ for time $t > 0$. Where is the particle at $t = 4$?

We know that velocity is the derivative of position. To find where the particle is at $t = 4$, we either need to find the position function (which we cannot do specifically with the given information) , or as we discovered above we can find the area under the velocity curve [since $t > 0$ and $v(t) > 0$, position is the same thing as distance traveled]

To find the area under the velocity curve, we need to partition the region into vertical strips. In the diagram at the right, notice that the region under the curve is partitioned into thin strips with bases of length $\frac{1}{4}$ and *curved* tops that slope upward from left to right. Unfortunately, we do not know how to find the area of such a shape. However, we can get a good approximation of it by finding the area of a suitable rectangle.



So how do we find a suitable rectangle? Let's find out.

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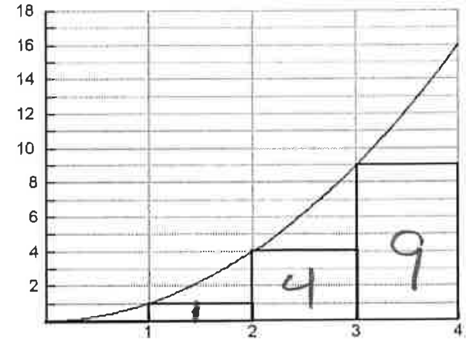
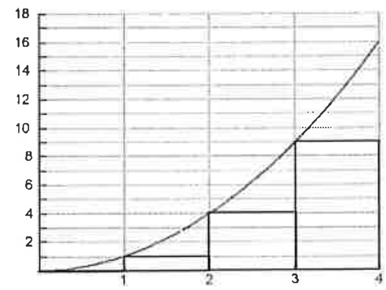
Using the Left Rectangular Approximation Method (LRAM) to find the area under $v(t) = t^2$, we let the height of our rectangles be the upper-left vertex of our rectangle. In the diagram at the right, the area from $0 \le t \le 4$ is partitioned using rectangles of base length 1. A length of 1 is not ideal (we would like it to be much smaller), but 1 will be easier for the purpose of our example.



Find the approximation of the area under the curve $v(t) = t^2$ using the LRAM and the rectangles in the diagram to the right. If $v(t) = t^2$ represents velocity (in meters/sec), what is the position of the particle at $t = 4$? What do you think of the accuracy of your estimation?

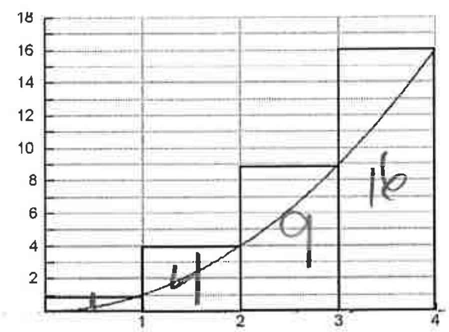


$1 + 4 + 9 = 14$

– The particle is 14 m from its starting point of $x = 0$.

– This estimate is not great (Lots of empty area)

To the right is an approximation of the area under the Right Rectangular Approximation Method (RRAM) to approximate the area under $v(t) = t^2$ from $0 \le t \le 4$. Find an approximation of the area using the RRAM. If $v(t) = t^2$ represents velocity (in meters/sec), what is the position of the particle at $t = 4$? What do you think of the accuracy of your estimation?



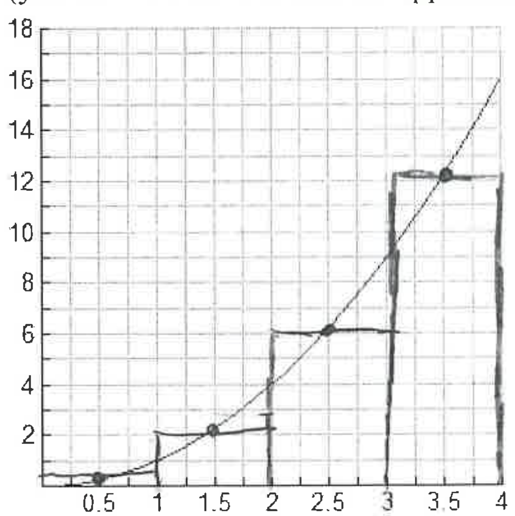
$1 + 4 + 9 + 16 = 30$

– The particle is 30 m from its starting point of $x = 0$.

– The estimate is an extreme overestimate.

In terms of the LRAM and the RRAM, what do you think is the actual area under the curve of $v(t) = t^2$ from $0 \le t \le 4$? We could clearly improve our estimate of area by making more rectangles. How do you think we can find a better approximation while *still using the same amount of rectangles*? Draw it below (you do NOT have to find the approximation)

Draw the rectangles so that the height is in the middle (midpt.)



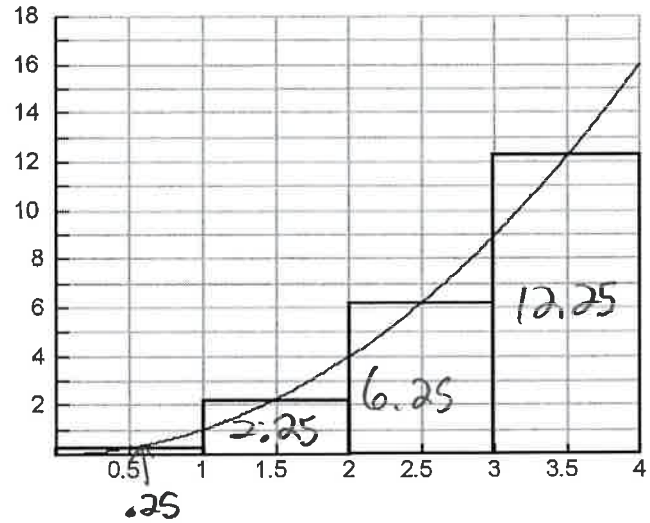
Will your approximation be an overestimate or an underestimate?

Answers will vary (Looks like underestimate.)

Explain, in mathematical terms, when a LRAM and a RRAM will be an overestimate or an underestimate.

The most accurate method for approximating the area under a curve with rectangles is the Midpoint Rectangular Approximation Method (MRAM). Using MRAM, the height of the rectangle is the *midpoint* of the selected intervals.

To the right is an approximation of the area under the MRAM to approximate the area under $v(t)=t^2$ from $0 \le t \le 4$. Find an approximation of the area using the MRAM. If $v(t)=t^2$ represents velocity (in meters/sec), what is the position of the particle at $t=4$? What do you think of the accuracy of your estimation?



$.25 + 2.25 + 6.25 + 12.25 = 21$

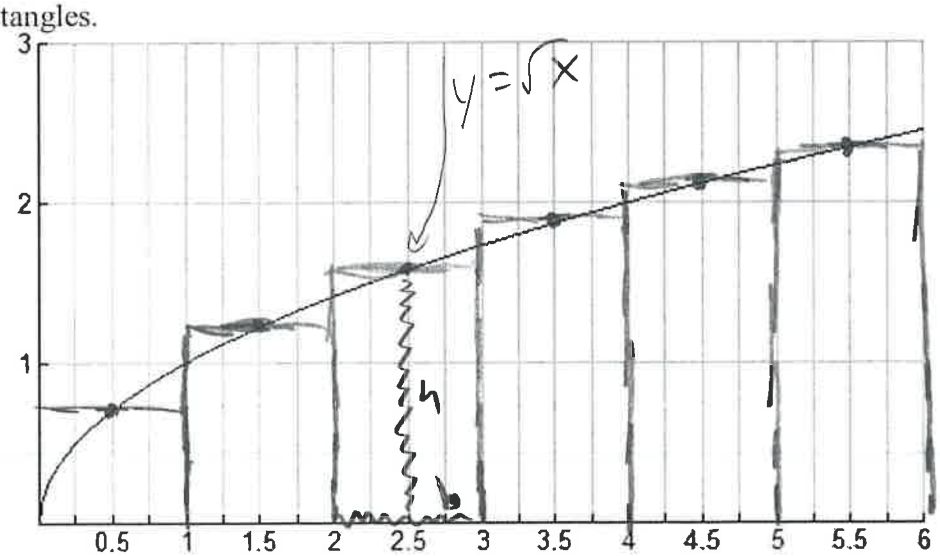
- The particle is 21 m from its starting point of $x=0$.

- This looks fairly accurate.

Note that as the size/number of rectangles grows (goes to infinity), the LRAM, RRAM, and MRAM *all will converge to the same limit* – which is the actual area. Partitioning the area into rectangles in this way is called finding **Riemann sums** – much more on this next section.

Practice

1. Estimate the area under the curve of $y=\sqrt{x}$ from $x=0$ to $x=6$ using the MRAM and 6 rectangles.



$A = 1 * \sqrt{.5} + 1 \cdot \sqrt{1.5} + 1 \cdot \sqrt{2.5} + 1 \cdot \sqrt{3.5} + 1 \cdot \sqrt{4.5} + 1 \cdot \sqrt{5.5}$

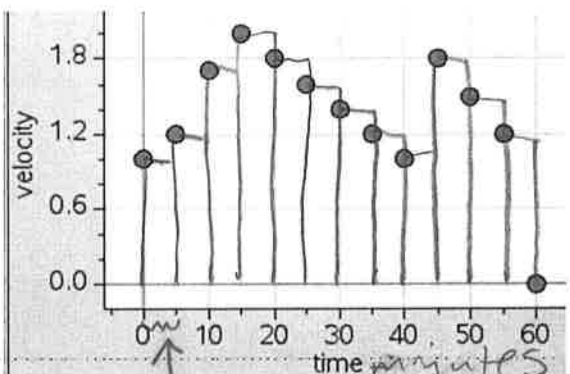
≈ 9.850

[Actual area ≈ 9.798]

Time (min)	Velocity (m/sec)	Time (min)	Velocity (m/sec)
0	1	35	1.2
5	1.2	40	1.0
10	1.7	45	1.8
15	2.0	50	1.5
20	1.8	55	1.2
25	1.6	60	0
30	1.4		

2. You are walking along the bank of a tidal river watching the incoming tide carry a bottle upstream. You record the velocity of the flow every 5 minutes for an hour, with the results shown in the table below. Estimate the distance upstream the bottle travels during that hour? Find the **(a)** LRAM and **(b)** RRAM estimates using 12 subintervals. Explain what your two answers tell you about the distance traveled by the bottle.

LRAM



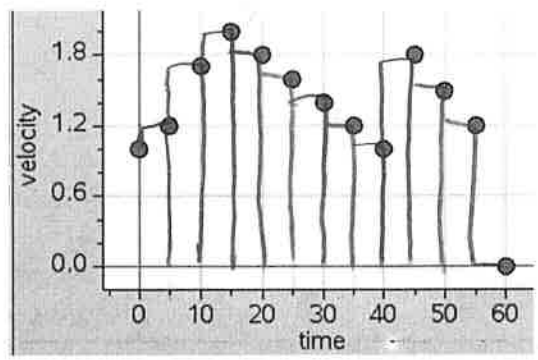
5 min = 300 sec

A=

$300(1+1.2+1.7+2+1.8+1.6+1.4+1.2+1+1.8+1.5+1.2)$

$\boxed{= 5220 \text{ m}}$

RRAM



A=

$300(1.2+1.7+2+1.8+1.6+1.4+1.2+1+1.8+1.5+1.2+0)$

$\boxed{= 4920 \text{ m}}$

Explanation: The bottle traveled somewhere between 4920 m & 5220 m.